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## Three manifestations of the pulsed harmonic potential

T J Newman and R K P Zia

Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

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**Abstract.** We consider, in turn, three systems being acted upon by a regularly pulsed harmonic potential (PHP). These are (i) a classical particle, (ii) a quantum particle, and (iii) a directed line. We contrast the mechanics of the first two systems by parametrizing their bands of stability and periodicity. Interesting differences due to quantum fluctuations are examined in detail. The fluctuations of the directed line are calculated in the two cases of a binding PHP, and an unbinding PHP. In the latter case there is a finite maximum line length for a given potential strength.

### 1. Introduction

Within the modelling of many problems in mechanics, the potential is taken to be time-independent, and sets the stage for the ensuing particle dynamics. Quite generally, however, the potential may have its own (parametric) dynamics: the stage is shifting beneath the actors' feet, so to speak. In quasistatic problems this parametric dynamics will be on such slow timescales that it may be safely ignored; or else, an adiabatic treatment may be used [1]. On the other hand, if the timescale is of exceedingly rapid character, it may be easier to forget the potential altogether, and model the parametric variation through some fast degrees of freedom. The Langevin equation represents a possible outcome of such a procedure [2]. There may be situations where neither the quasistatic nor the Langevin approaches is appropriate. In these cases, one is obliged to face the time-dependence of the potential head-on—this leads to real analytic difficulties, especially in the quantum mechanical case, where one must abandon the notion of eigenstates, and tackle the time-dependent Schrödinger equation directly.

The analytic challenge of explicitly time-dependent mechanics is great, and exactly solvable cases are valuable for providing basic insights. The harmonic potential is widely studied in many areas of physics, due partly to its inherent solvability. It is also one of the most ubiquitous potentials in Nature, due to the existence of near-equilibrium states. It is for these reasons that the classical and quantum mechanics of the explicitly time-dependent harmonic potential have been studied for many years [1, 3–7].

One of the most extreme limits within this class of problems is that of the pulsed harmonic potential (PHP). That is to say, the potential exists for extremely short instants, between which there is no potential whatsoever. This may be represented by

$$V(\mathbf{x}, t) = v(\mathbf{x}) \sum_n \delta(t - \tau_n) \quad (1)$$

where in particular

$$v(\mathbf{x}) = (\lambda/2)x^2 \quad (2)$$

and for regular pulsing  $\tau_n = n\tau$ . (An alternative terminology has arisen in the field of quantum chaos [8], in which one speaks of the system being ‘kicked’ by such a potential.) Apart from more obvious manifestations of such a potential (in which an experimenter externally pulses a system with some form of trapping potential), one can envisage such a situation arising in the frame of reference of a rapidly moving particle as it regularly passes through regions within which a static harmonic potential exists.

To our knowledge there is no complete treatment of this system. In fact, we suspect that its apparent simplicity may have persuaded workers to add complicating features. We are aware of similar harmonic models in the field of quantum chaos [9, 10], but these are typically considered on a toroidal phase space in order to make closer contact to classically chaotic systems. In such a case, it is known that only periodic orbits exist in the quantum case, whereas the classical system may be chaotic depending on the curvature of the potential. We are content to study the system on the full phase space. A great deal of effort has also been directed towards the problem of a ‘kicked harmonic oscillator’ [11], by which is meant a static harmonic potential periodically pulsed with some spatially periodic potential. This is used to model the behaviour of a trapped particle under the action of a laser.

To give a unifying flavour to this work, we shall study three manifestations of the problem. In section 2 we consider a classical particle in a PHP and parametrize the region of stability, along with its associated periodic and quasiperiodic dynamics. In sections 3–5 we consider a quantum particle in a PHP. Sections 3 and 4 are concerned with a wavepacket centred at the origin, which has no classical analogue. Two different analytic formulations of the problem are presented, each with their advantages in application. We parametrize the stability and periodicity of the dynamics (including a bizarre cycle with period  $\tau$ ) and also examine the limit of  $\tau \rightarrow 0$  which we compare to the static harmonic potential. In section 5 we examine an off-centred wavepacket. Contact is made with the classical system via Ehrenfest’s theorem, and we also show that the expectation value of the Hamiltonian splits neatly into two pieces which are, respectively, purely quantum and classical in origin. In section 6 we study the third manifestation of a PHP: namely, a directed line in thermal equilibrium with a set of planar harmonic potentials. We determine the asymptotic transverse fluctuations of the line for binding potentials, and the finite maximal length of the line for unbinding potentials. We end the paper with section 7, in which we give a detailed summary of our results, along with some general conclusions.

## 2. Classical particle in PHP

We consider a classical particle of mass  $m$  in a PHP, which is equivalent to the periodic impulsive force

$$F(x, t) = -\lambda x \sum_{n=1}^{\infty} \delta(t - n\tau). \quad (3)$$

The particle will suffer a discontinuous change in momentum with a period of  $\tau$ . In the intervening intervals, the particle changes its position with a constant velocity. It is sufficient to describe the particle’s trajectory by

$$x_n \equiv \lim_{\epsilon \rightarrow 0} x(n\tau - \epsilon) \quad p_n \equiv \lim_{\epsilon \rightarrow 0} p(n\tau - \epsilon). \quad (4)$$

These quantities satisfy the difference equations

$$\begin{aligned} x_{n+1} &= x_n + (\tau/m)p_{n+1} \\ p_{n+1} &= p_n - \lambda x_n. \end{aligned} \quad (5)$$

We must also specify the initial conditions for the particle:  $x(0)$  and  $p(0)$ . The initial values for the difference equations may then be given as  $p_1 = p(0)$ , and  $x_1 = x(0) + (\tau/m)p(0)$ .

It is convenient to rescale the momentum to  $\rho_n = (\tau/m)p_n$ , so that the difference equations now take the form

$$\begin{aligned} x_{n+1} &= x_n + \rho_{n+1} \\ \rho_{n+1} &= \rho_n - \xi x_n \end{aligned} \tag{6}$$

where

$$\xi = \lambda\tau/m \tag{7}$$

is the dimensionless coupling to the potential. It is a simple matter to eliminate one set of the difference functions ( $\{x_n\}$  say), to give the single second-order difference equation

$$\rho_{n+2} = \beta\rho_{n+1} - \rho_n \tag{8}$$

where  $\beta = 2 - \xi$ . We supply the two required initial data  $\rho_1$  and  $\rho_0 \equiv \rho_1 + \xi(x_1 - \rho_1)$ . (One may also proceed by recasting equation (6) in matrix form and determining the dynamics and stability of the system from the associated eigenvalues. We shall use the second-order difference equation in order to make closer contact with the analysis of the quantum system in the following section.)

This difference equation may be easily solved by introducing the generating function

$$R(z) = \sum_{n=0}^{\infty} z^n \rho_n \tag{9}$$

which may be inverted via the contour integral,

$$\rho_n = \frac{1}{2\pi i} \int_C \frac{dz}{z^{n+1}} R(z) \tag{10}$$

where  $C$  encircles the origin anticlockwise (with a radius chosen small enough so as not to enclose any singularities bar the pole at the origin).

Summing the difference equation for  $\rho$  with a weight of  $z^n$  yields the following expression for the generating function:

$$R(z) = \frac{(1 - \beta z)\rho_0 + z\rho_1}{z^2 - \beta z + 1}. \tag{11}$$

This function has two simple poles located at  $\alpha_1$  and  $\alpha_2$  where

$$\alpha_{1,2} = \frac{\beta}{2} \pm \frac{1}{2}(\beta^2 - 4)^{1/2}. \tag{12}$$

For  $|\beta| > 2$  the poles lie on the real axis and one of them has a modulus less than unity. Referring to equation (10), we see this implies that  $|\rho_n|$  grows unboundedly with increasing  $n$ . Thus, stable evolution of the particle is only possible for  $|\beta| \leq 2$  (corresponding to  $0 \leq \xi \leq 4$ ), which we now examine in more detail. Defining a parameter  $\phi$  via  $\beta = 2 \cos \phi$ , we have  $\alpha_{1,2} = e^{\pm i\phi}$ . We evaluate the contour integral in equation (10) by noting that the contour  $C$  may be deformed around the singularities away from the origin such that  $\int_C = -\int_{\text{sing}}$ . In other words, the required integral along  $C$  is equal to minus the residues from the two poles. Evaluating these residues, and performing some algebraic manipulations, we arrive at the result

$$\rho_{n+1} = \frac{1}{\sin \phi} [\rho_1 \sin(n + 1)\phi - \rho_0 \sin n\phi]. \tag{13}$$

It is clear from the above expression that some form of cyclic behaviour with period  $n\tau$  will occur when the value of  $\xi$  (and thus  $\beta$ ) is adjusted so that  $\sin n\phi = 0$ . In this case we have  $\phi = M\pi/n$ , where  $M = 1, \dots, [n/2]$ , and consequently,  $\rho_{n+k} = (-1)^M \rho_k$ . From equation (6) it is easy to check that such a value of  $\phi$  also implies  $x_{n+k} = (-1)^M x_k$ .

Let us classify two types of periodic motion: PMI( $n$ )—a motion for which all measurable quantities take on the same values with a period of  $n\tau$ ; and PMII( $n$ )—a motion for which only the energy of the system has a period of  $n\tau$ . Now, the energy (between pulses) is simply given by

$$E_n = \frac{m}{2} \left( \frac{\rho_n}{\tau} \right)^2. \quad (14)$$

Thus, PMII( $n$ ) occurs for *any* value of  $\phi$  satisfying  $\sin n\phi = 0$ . However, PMI( $n$ ) only occurs for values of  $\phi$  which are an *even* multiple of  $\pi/n$ . So the simplest PMII motion occurs for  $n = 2$ , which corresponds to  $\xi = 2$ . However, the simplest PMI motion occurs for  $n = 3$  with  $M = 2$ , and corresponds to  $\xi = 3$ . We stress that these periodic motions exist once the parameter  $\xi$  is tuned to an appropriate value, regardless of the initial data  $(x(0), p(0))$ .

As a final remark, we note that there exists one special PMI motion which has a period of  $2\tau$ . This may be seen directly from the first-order difference equations (6). Such a motion is possible if one tunes  $\xi = 4$  and adjusts the initial data such that  $\rho_1 = 2x_1$ .

### 3. Quantum particle in PHP: 1

In this and the next two sections we shall examine the evolution of a Gaussian wavepacket under the influence of a PHP. We shall highlight the similarities of the mean motion to the classical dynamics described in section 2, as well as some subtle effects which are of a purely quantum origin. In this section we shall examine the evolution of a wavepacket centred at the origin, and we shall use direct evaluation of Gaussian integrals to arrive at our results. In the next section, we study the same problem, but with the aid of Fourier decomposition. Both methods yield the same results, but in surprisingly different formats, which are individually suited to the calculation of different quantities. The final section of the three is concerned with the evolution of an off-centred wavepacket (which has non-zero expectation values for position and momentum and may therefore be compared to the classical case).

Our starting point is Schrödinger's equation for the wavefunction  $\psi(x, t)$ , with the potential given as in equations (1) and (2):

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \partial_x^2 \psi + \frac{\lambda}{2} x^2 \sum_{n=1}^{\infty} \delta(t - n\tau) \psi. \quad (15)$$

As an initial condition we take a centred Gaussian

$$\psi(x, 0) = (2B^R/\pi)^{1/4} \exp[-Bx^2] \quad (16)$$

where  $B$  is complex, and  $B^R \equiv \text{Re}[B] > 0$ . This choice is made on the grounds of simplicity, but we can imagine preparing such a state from the lowest eigenstate of a static harmonic potential. The evolution of the wavefunction between pulses is simply free particle propagation. Therefore we can describe the dynamics by the set of functions  $\{\psi_n(x)\}$ , where

$$\psi_n(x) = \lim_{\epsilon \rightarrow 0} \psi(x, n\tau - \epsilon). \quad (17)$$

It is not a trivial matter to determine the change in the wavefunction due to a pulsed potential. We refer the reader to [12] for a full discussion of this point (in the statistical mechanics

context). The result is that the wavefunction suffers a discontinuity in phase. Thus, the probability density of the particle is unchanged in the immediate temporal vicinity of the pulse. With regard to the general pulsed potential given in equation (1), the wavefunction immediately after the pulse is given by

$$\psi_n^+(x) \equiv \lim_{\epsilon \rightarrow 0} \psi(x, n\tau + \epsilon) = \psi_n \exp\left[-\frac{iv(x)}{\hbar}\right] \quad (18)$$

which has a very natural form when viewed from the path-integral perspective [13]. In the present case of a PHP we have

$$\psi_n^+(x) = \psi_n \exp\left[-\frac{i\lambda x^2}{2\hbar}\right]. \quad (19)$$

The free particle propagation between pulses may be written as

$$\psi_{n+1}(x) = \int dx' G(x - x', \tau) \psi_n^+(x') \quad (20)$$

where the Green function has the familiar form [14]

$$G(x, t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left[\frac{imx^2}{2\hbar t}\right]. \quad (21)$$

Combining equations (19) and (20) yields the iteration rule (which resembles a transfer matrix in the statistical mechanics context) for the functions  $\{\psi_n(x)\}$ ; namely,

$$\psi_{n+1}(x) = \int dx' G(x - x', \tau) \exp\left[-\frac{i\lambda x'^2}{2\hbar}\right] \psi_n(x'). \quad (22)$$

It is convenient to rescale  $x \rightarrow y = x/b$ , where  $b = (\hbar\tau/m)^{1/2}$  and define the dimensionless parameters,  $\xi = \lambda\tau/m$  (cf equation (7) in section 2), and  $\eta = 2Bb^2$ . The iteration rule now takes the form

$$\psi_{n+1}(y) = (2\pi i)^{-1/2} \int dy' \exp\left[\frac{i}{2}(y - y')^2 - \frac{i\xi}{2}y'^2\right] \psi_n(y') \quad (23)$$

and from equation (16), the initial wavefunction is given by

$$\psi(y, 0) = (\eta^R/\pi b^2)^{1/4} \exp[-\eta y^2/2] \quad (24)$$

where  $\eta^R = \text{Re}[\eta]$ .

We shall concentrate on calculating two important physical quantities: the probability density  $P(x, t) = |\psi(x, t)|^2$  and the expectation value of the Hamiltonian (between pulses). The latter is defined at the moment prior to the pulse:

$$E_n = \int dx \psi_n(x)^* \left[-\frac{\hbar^2}{2m} \partial_x^2\right] \psi_n(x) \quad (25)$$

but is constant for the duration of the interval between two adjacent pulses.

The iteration rule (23) clearly shows that the wavefunction will have a Gaussian form for all times, given that its initial form is chosen to be a Gaussian. Thus, we write the general form for  $\psi_n$  (in the unscaled  $x$  coordinate) as

$$\psi_n(x) = A_n \exp\left[-\sigma_n \frac{x^2}{2b^2}\right] \quad (26)$$

where  $A_n$  and  $\sigma_n$  are complex numbers. The probability density is given by

$$P_n(x) = |A_n|^2 \exp[-C_n x^2/b^2] \quad (27)$$

where  $C_n = (\sigma_n + \sigma_n^*)/2 = \pi b^2 |A_n|^4$ , the latter equality following from normalization. From equations (25) and (26) one also has

$$E_n = \frac{\hbar}{2\tau} \frac{|\sigma_n|^2}{(\sigma_n + \sigma_n^*)}. \quad (28)$$

It is useful to define

$$I_n \equiv \int dx \psi_n(x) = A_n (2\pi b^2 / \sigma_n)^{1/2} \quad (29)$$

which may be inverted to yield

$$\sigma_n = 2\pi b^2 (A_n / I_n)^2. \quad (30)$$

Therefore, we may determine all the quantities of interest by evaluating  $A_n$  (the value of the wavefunction at the origin), and  $I_n$  (the spatial integral of the wavefunction).

As a first step in the evaluation of these two quantities, let us explicitly iterate the function  $\psi_n$  back to the initial condition. Using equations (23) and (24) we have

$$\psi_n(y) = (\eta^R / \pi b^2)^{1/4} (2\pi i)^{-n/2} \int dy_{n-1} \dots \int dy_0 \exp[-\frac{1}{2} y_l M_{lm}^{(n)} y_m] \quad (31)$$

where the  $n \times n$  matrix  $M^{(n)}$  has diagonal elements  $M_{00}^{(n)} = \eta - i$ ,  $M_{ll}^{(n)} = -i\beta$ , and off-diagonal elements  $M_{lm}^{(n)} = i$  for  $|l - m| = 1$ , and  $M_{lm}^{(n)} = 0$  otherwise.

Thus, the wavefunction at the origin is given by

$$A_n = (\eta^R / \pi b^2)^{1/4} i^{-n/2} q_n^{-1/2} \quad (32)$$

where  $q_n \equiv \det M^{(n)}$ . Also, integrating the above  $n$ -fold integral over  $y$ , we find

$$I_n = (2\pi b^2)^{1/2} (\eta^R / \pi b^2)^{1/4} i^{-n/2} (-iq_n + q_{n-1})^{-1/2}. \quad (33)$$

So we may describe the entire dynamics from the set of determinants  $\{q_n\}$ . Before explicitly calculating these functions, we shall first express the physical quantities of interest in terms of  $\{q_n\}$ . From equation (30) we may combine the above two expressions to give (for  $n \geq 1$ )

$$\sigma_n = -i \left( \frac{q_n + iq_{n-1}}{q_n} \right). \quad (34)$$

We therefore have the explicit form for the wavefunction (for  $n \geq 1$ ):

$$\psi_n(x) = (\eta^R / \pi b^2)^{1/4} i^{-n/2} q_n^{-1/2} \exp \left\{ \frac{ix^2}{2b^2} \left( \frac{q_n + iq_{n-1}}{q_n} \right) \right\}. \quad (35)$$

The wavefunction at intervening times may be easily found by propagating the above form with the Green function (21). Straightforward integration yields

$$\begin{aligned} \psi(x, (n-1)\tau + \theta\tau) &= (\eta^R / \pi b^2)^{1/4} i^{-n/2} [\theta q_n - i(1 - \theta q_{n-1})]^{-1/2} \\ &\times \exp \left\{ \frac{ix^2}{2b^2} \left[ \frac{q_n + iq_{n-1}}{\theta q_n - i(1 - \theta q_{n-1})} \right] \right\} \end{aligned} \quad (36)$$

where  $\theta \in [0, 1]$ .

We may determine the probability density just prior to pulsing (i.e. the function  $P_n(x)$ ) either from our knowledge of  $A_n$  (cf equation (27)), or by simply taking the modulus squared of  $\psi_n$  as given in equation (35). The results of these procedures, although strictly identical, are not obviously so, since their equivalence requires the following identity to hold:

$$\text{Re}[q_n q_{n-1}^*] = \eta^R. \quad (37)$$

The proof of this statement will be given shortly. The explicit form of the probability density is given by (for general times)

$$P(x, (n - 1)\tau + \theta\tau) = \frac{(\eta^R/\pi b^2)^{1/2}}{|\theta q_n - i(1 - \theta)q_{n-1}|} \exp \left\{ -\frac{x^2}{b^2} \frac{\eta^R}{|\theta q_n - i(1 - \theta)q_{n-1}|^2} \right\}. \quad (38)$$

The important information contained in the probability density is the temporal evolution of the width of the Gaussian wavepacket. Just prior to pulsing we have  $P_n \sim \exp[-x^2/\gamma_n^2]$ , where the width  $\gamma_n$  is simply given by

$$\gamma_n = |q_n| \gamma_0 \quad (39)$$

where the initial width  $\gamma_0 = b/\sqrt{\eta^R}$ . Finally, for the expectation value of the Hamiltonian between pulses (i.e. the mean energy), we may derive from equations (28), (34) and (37) the result

$$E_n = \frac{\hbar}{4\tau\eta^R} |q_n + iq_{n-1}|^2. \quad (40)$$

We now turn to the evaluation of the set of determinants  $\{q_n\}$ . From the definition of the matrix  $M^{(n)}$  it is straightforward to derive the following iteration rule

$$q_{n+2} = -i\beta q_{n+1} + q_n \quad (41)$$

where  $\beta = 2 - \xi$ . This is very similar to the second-order difference equation (8) that we derived previously for the rescaled momenta in the classical system. In the present case, the initial data are  $q_0 = 1$ , and  $q_1 = \eta - i$ . Before proceeding to solve equation (41), let us first prove the assertion (37). Multiplying through the difference equation (41) by  $q_{n+1}^*$  yields

$$q_{n+2}q_{n+1}^* = -i\beta|q_{n+1}|^2 + q_{n+1}^*q_n. \quad (42)$$

Now the first term on the right-hand side is purely imaginary, thus the bilinear object  $\text{Re}[q_nq_{n-1}^*]$  is independent of  $n$  and therefore equal to

$$\text{Re}[q_nq_{n-1}^*] = \text{Re}[q_1q_0^*] = \text{Re}[\eta - i] = \eta^R \quad (43)$$

as required.

The solution of the difference equation (41) may be found by use of a generating function just as in the classical system. Bounded evolution of  $|q_n|$  is again limited to the parameter regime  $|\beta| \leq 2$ , which corresponds to  $0 \leq \xi \leq 4$ . It is convenient to define the parameter  $\phi$  via

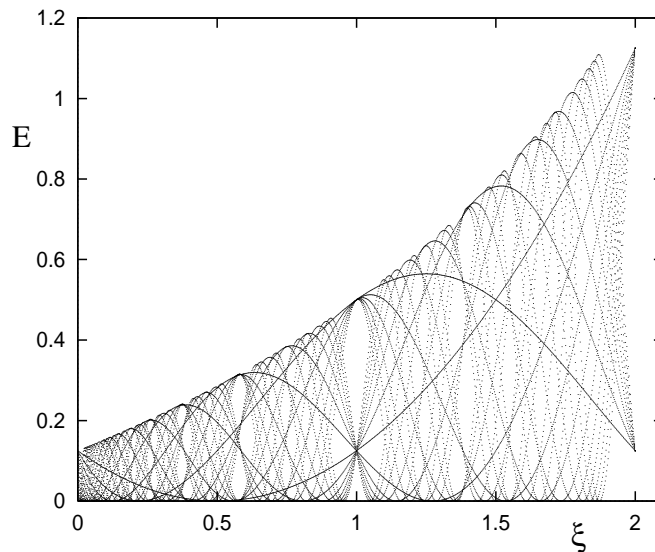
$$\cos \phi = \beta/2 = 1 - \xi/2 \quad (44)$$

as in the classical system. The explicit form for  $q_n$  is found to be

$$q_n = \frac{(-i)^n}{\sin \phi} [i(\eta - i) \sin n\phi - \sin(n - 1)\phi]. \quad (45)$$

It is very tempting to make a direct connection between these results, and those for the classical system. However, in doing so we would lose some of the subtleties contained within. Consider first, the present quantum dynamics has no classical analogue, since the expectation values of the position and momentum remain zero for all times. Second, there is a difference between the types of periodic motion in the quantum system considered here, and the classical system. In the latter we found it useful to classify periodic behaviour into two categories: PMI( $n$ ) (all physical quantities having period  $n\tau$ ) and PMII( $n$ ) (the energy having period  $n\tau$ ). The condition for motion of the first class was  $\phi = M\pi/n$ , with  $M$  even; whereas the condition for the second class was  $\phi = M\pi/n$  with  $M$  odd or even. In





**Figure 1.** Energy portrait for a classical particle of unit mass in a PHP for  $0 \leq \xi \leq 2$ , and initial conditions  $x(0) = 1$  and  $\rho_1 = \frac{1}{2}$ .

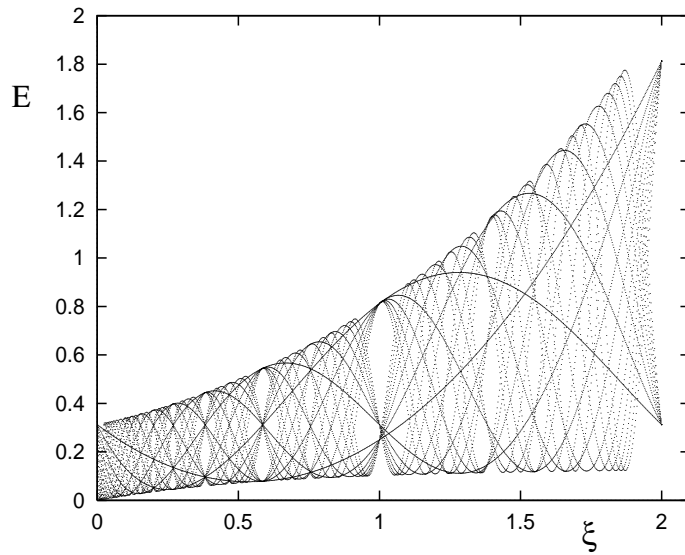
the present case of quantum motion, it is clear from equation (45) that periodic behaviour is to be expected for  $\sin n\phi = 0$  which corresponds to  $\phi = M\pi/n$  with  $M = 1, \dots, [n/2]$ . However, apart from setting the scale of  $\phi$ , the integer  $M$  plays no other role, since it appears in the wavefunction as a constant phase factor  $e^{-iM\pi/2}$ . So there is no distinction between PMI and PMII motions in the quantum dynamics of a centred Gaussian wavefunction.

It is of interest to compare the energy portraits for the classical and quantum motions. We scan through values of  $\xi \in (0, 2)$  and follow the evolution of the energy (as given by equations (14) and (40) respectively) for 20 or so iterations, all of which values are plotted. We refer the reader to figures 1–5 for the classical (with typical values of  $x(0)$  and  $p(0)$ ) and quantum (with varying initial parameter  $\eta$ ) portraits. Despite the similarity in the values of  $\xi$  at which the periodic orbits occur, we see a distinct difference in the bounding curves  $E_{\min}(\xi)$  and  $E_{\max}(\xi)$  for the minimum and maximum energies. In the classical case,  $E_{\min}$  is essentially zero and  $E_{\max}$  is a monotonically increasing function of the coupling  $\xi$ . In the quantum case,  $E_{\min}$  maintains a distinct gap from zero, and the difference  $E_{\max} - E_{\min}$  is non-monotonic with  $\xi$ . It would be of interest to determine the analytic properties of these bounding curves, but this is beyond the scope of the present work.

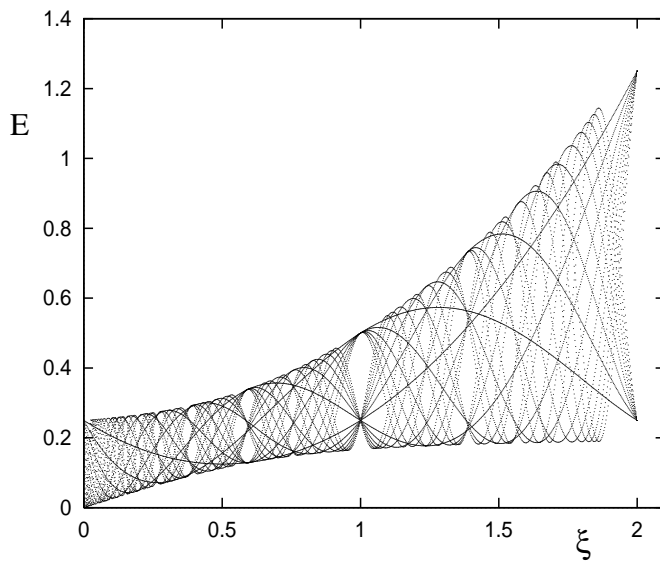
We shall make a more physical connection between the quantum and classical motions in section 5 when we consider an off-centred wavefunction which allows the particle to maintain non-zero expectation values of both position and momentum. First, however, we present an alternative treatment of the centred Gaussian wavefunction, which will allow us to study some other properties of this system more easily.

#### 4. Quantum particle in PHP: 2

Although the calculations of the previous section were reasonably straightforward, they still required the evaluation of cumbersome  $n$ -fold Gaussian integrals. It might be hoped that a simpler derivation of the results is possible, since the wavefunction evolves by free wave



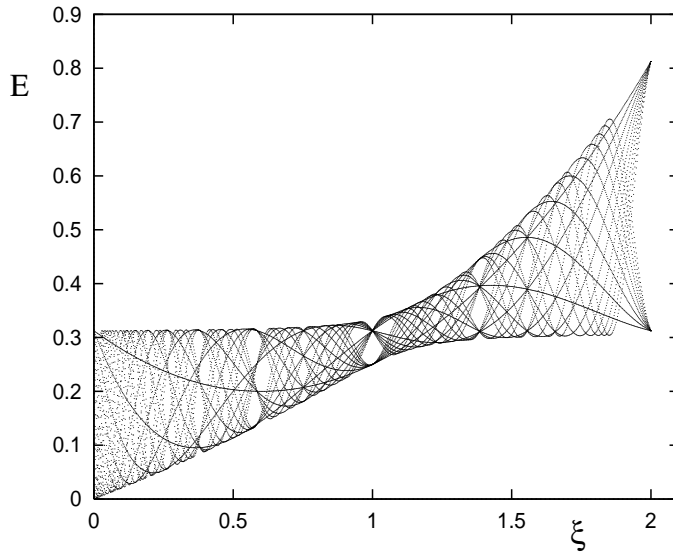
**Figure 2.** Energy portrait for a quantum particle in a PHP for  $0 \leq \xi \leq 2$  with  $\hbar = \tau = 1$ , and  $\eta = 1 - i/2$ .



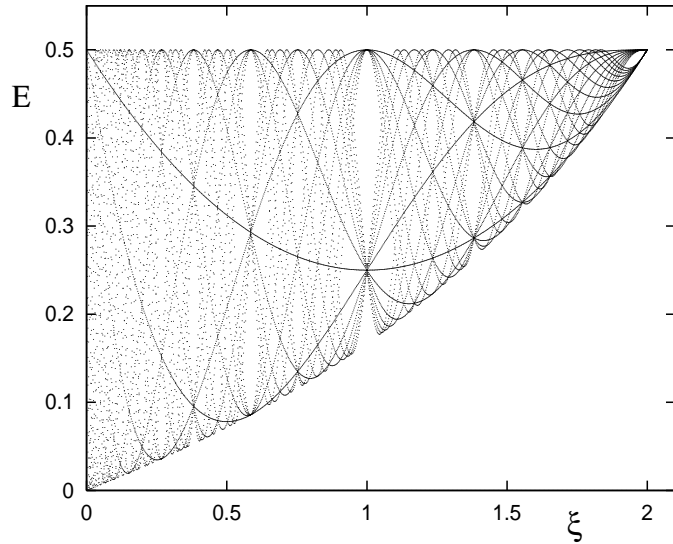
**Figure 3.** As figure 2 with  $\eta = 1$ .

propagation between pulses, which may be handled more easily in Fourier space. The purpose of this section is to derive an iteration rule for the wavefunction using this Fourier method. The surprise is that the rule (or difference equation) turns out to be nonlinear and of first order, in contrast to the linear second-order rule (41) derived above. The two iteration rules must yield the same results of course; but, as we shall see, they are non-trivially related.

To proceed, we restate that given our initial wavefunction is Gaussian, we can expect



**Figure 4.** As figure 2 with  $\eta = 1 + i/2$ .



**Figure 5.** As figure 2 with  $\eta = 1 + i$ . Note the 1-cycle at  $\xi = 2$ .

the wavefunction at all subsequent times to retain a Gaussian form. So, we write

$$\psi_n(x) \sim \exp\left[-\sigma_n \frac{x^2}{2b^2}\right] \tag{46}$$

where we have omitted the prefactor. The wavefunction at the instant after the pulse is given by equation (19) which we rewrite here as

$$\psi_n^+(x) \sim \exp\left[-(\sigma_n + i\xi) \frac{x^2}{2b^2}\right]. \tag{47}$$

The next stage of the evolution is free wave propagation, which is most easily expressed in Fourier space:  $\tilde{\psi}(k, t + \tau) = \tilde{\psi}(k, t) \exp[-ib^2k^2/2]$ . Now the Fourier transform of  $\psi_n^+$  takes the form

$$\tilde{\psi}_n^+(k) \sim \exp\left[-\frac{k^2b^2}{2(\sigma_n + i\xi)}\right] \tag{48}$$

and so

$$\tilde{\psi}_{n+1}(k) \sim \exp\left[-\frac{k^2b^2}{2}\left(\frac{1}{\sigma_n + i\xi} + i\right)\right]. \tag{49}$$

Finally, inverse transforming the above expression we arrive at

$$\psi_{n+1}(x) \sim \exp\left[-\sigma_{n+1}\frac{x^2}{2b^2}\right] \tag{50}$$

where

$$\frac{1}{\sigma_{n+1}} = \frac{1}{\sigma_n + i\xi} + i. \tag{51}$$

This iteration rule requires only one piece of initial datum. Evolving the initial wavefunction to  $t = \tau - 0$ , we have

$$\sigma_1 = \eta/(1 + i\eta). \tag{52}$$

Given this nonlinear iteration rule, one might expect the system to have some kind of non-trivial (chaotic) dynamics. However, with the benefit of hindsight, we know from section 3 that in fact the system has either periodic or quasiperiodic behaviour (in the stable regime  $0 \leq \xi \leq 4$ ). This simpler behaviour is not apparent from the nonlinear rule for  $\{\sigma_n\}$ , but we can prove it is the case by connecting this rule to the linear second-order rule for  $\{q_n\}$  given in equation (41).

Referring to (34), we have the relation (for  $n \geq 1$ )

$$\sigma_n = -i\left(\frac{q_n + iq_{n-1}}{q_n}\right). \tag{53}$$

Substituting this result into the nonlinear iteration rule (51), we find equality only if  $q_n$  satisfies the linear iteration rule (41), as required for consistency. In the absence of hindsight, relation (53) would be seen as a rather remarkable linearizing transformation. In order to probe the general (quasi)-periodic behaviour of this system, the linear iteration rule is the description of choice. However, there are two other aspects of this system which are much more easily described by the nonlinear rule given above.

The first of these is the existence of a special periodic motion with period  $\tau$ . We may identify this by simply demanding that  $\sigma_{n+1} = \sigma_n (= \sigma)$  in equation (51). This yields a quadratic equation for  $\sigma$  with (normalizable) solution

$$\sigma = \frac{1}{2}[\xi(4 - \xi)]^{1/2} - \frac{i\xi}{2}. \tag{54}$$

Now, the initial value of  $\sigma$  is set by equation (52). So this special ‘1-cycle’ is only possible (for a given value of  $\xi \in [0, 4]$ ) for a special value  $\bar{\eta}$  of the parameter  $\eta$  (which, we remind the reader, describes the initial Gaussian wavefunction). We may invert (52) for  $\bar{\eta}$  to find

$$\bar{\eta} = \sin \phi + i(1 - \cos \phi) = 2 \sin(\phi/2) \exp(i\phi/2) \tag{55}$$

where we have used  $\phi$  in place of  $\xi$ , as defined in equation (44). An example of this 1-cycle is shown in figure 5. This 1-cycle is a purely quantum effect, since the classical system

can only have cycles of period  $\geq 2\tau$  for obvious reasons. One might term this motion ‘the sound of one hand clapping.’

The second aspect of the quantum system which may be extracted more easily from the nonlinear iteration is the behaviour of the system as the time between pulses is taken to zero. Before taking this limit, it is important to scale out  $\tau$  in other quantities. Now, we shall compare the system (in the limit of  $\tau \rightarrow 0$ ), to the quantum mechanics of the static harmonic potential. For the latter system, the potential is taken to be  $V_s(x) = \kappa x^2/2$ . The time-averaged potential of the PHP is  $\bar{V}(x) = \lambda x^2/2\tau$ . Thus we shall take the limit  $\tau \rightarrow 0$  and  $\lambda \rightarrow 0$  with the ratio  $\lambda/\tau \rightarrow \kappa$ . It is also convenient to define  $v_n = \sigma_n/\tau$ . Then the wavefunction just prior to pulsing is given by (cf equation (46))

$$\psi_n(x) \sim \exp\left[-\frac{m}{2\hbar}v_n x^2\right]. \quad (56)$$

From (51), the iteration rule for  $v_n$  takes the form

$$\frac{1}{v_{n+1}} = \frac{1}{v_n + i\lambda/m} + i\tau. \quad (57)$$

Taking the  $\tau \rightarrow 0$  limit as prescribed above, the above iteration rule tends to the first-order differential equation for  $v(t)$ :

$$\frac{dv}{dt} = i\left(\frac{\kappa}{m} - v^2\right). \quad (58)$$

Now, the ground state wavefunction of the static harmonic potential is a simple Gaussian  $\varphi(x) \sim \exp(-m\omega_s x^2/2\hbar)$ , where the oscillator frequency  $\omega_s = \sqrt{\kappa/m}$ . Referring to the differential equation for  $v(t)$ , we see that  $v = \omega_s$  is a fixed point of the dynamics. In other words, if we initialize the wavefunction to be the ground state wavefunction of the static harmonic potential, then the wavefunction will be completely unaffected by the PHP in the limit of  $\tau \rightarrow 0$ . We stress that there will be no dynamical evolution whatsoever. This is in contrast to the wavefunction of a truly static harmonic potential, which if prepared in the ground state, will still have a dynamically evolving phase factor  $e^{-i\omega_s t/2}$ . The pulsing of the potential essentially resets the clock of the wavefunction such that the dynamical phase is ‘stuck’ at  $t = 0$ . This effect may have important consequences for numerical integration of the time-dependent Schrödinger equation. If one places the equation on a discrete temporal grid, then one is pulsing the potential. From the present example, we see that a pulsed potential (on however fine a grid) cannot mimic a static potential. The dynamical phase information is irretrievably lost. Subjecting a system to very high frequency pulses was also studied recently in the context of controlling decoherence [15].

One can examine the dynamics of the PHP via the differential equation (58) in more detail. For instance, one can examine the evolution of a Gaussian wavepacket which is not tuned to be  $\varphi(x)$ . Let us restrict ourselves to an initial condition for  $\psi$  which is real, and therefore completely described by the width of the wavepacket, which we denote by  $l_0$ , and which is related to an effective frequency  $\omega = \hbar/ml_0^2$ . A straightforward solution of (58) yields

$$\psi(x, t) \sim \exp\left[-\frac{x^2}{2l_0^2} \left(\frac{1 + (i/r)\tan(\omega t)}{1 + ir\tan(\omega t)}\right)\right] \quad (59)$$

where  $r = \omega/\omega_s$ . Consequently, the probability density  $P(x, t) \sim \exp(-x^2/\gamma^2)$  with

$$\gamma(t) = l_0 \left(\frac{1 + r^2 \tan^2(\omega t)}{1 + \tan^2(\omega t)}\right). \quad (60)$$

The expectation value of the kinetic energy is given by

$$E_{\text{kin}}(t) = \frac{\hbar\omega}{4} \left( \frac{1 + (1/r)^2 \tan^2(\omega t)}{1 + \tan^2(\omega t)} \right). \quad (61)$$

In the limit of  $\tau \rightarrow 0$  there is also an effective potential energy (from averaging over the many pulses in a given small time interval). The expectation value of the potential energy may be found from  $\int dx \psi^*(\kappa x^2/2)\psi$ , with  $\psi(x, t)$  given by (59) above. One finds

$$E_{\text{pot}}(t) = \frac{\hbar\omega}{4r^2} \left( \frac{1 + r^2 \tan^2(\omega t)}{1 + \tan^2(\omega t)} \right). \quad (62)$$

The total energy is then found to be

$$E_{\text{tot}}(t) = \frac{\hbar\omega_s}{4} \left( r + \frac{1}{r} \right) \quad (63)$$

which is independent of time as expected.

### 5. Quantum particle in PHP: 3

In this third and final section concerning quantum dynamics in a PHP, we shall consider an initial wavefunction which is off-centred. Thus the expectation values of the position and momentum of the particle will be non-zero, and we may make closer contact between the quantum PHP and its classical counterpart. We shall utilize the iteration rules from both sections 3 and 4.

We begin by deriving the first-order iteration rules using the method described in section 4. Consider an off-centred Gaussian at the moment prior to the  $n$ th pulse:

$$\psi_n \sim \exp \left[ -\sigma_n \frac{x^2}{2b^2} + d_n \frac{x}{b} \right]. \quad (64)$$

Following the treatment for the centred Gaussian (cf equations (46)–(50)) we find that the iteration rule for  $\{\sigma_n\}$  is unchanged from the form given in equation (51), and the rule for  $\{d_n\}$  is

$$d_{n+1} = d_n \frac{\sigma_{n+1}}{(\sigma_n + i\xi)}. \quad (65)$$

With the aid of (51) and (53) we may rewrite this as

$$d_{n+1}q_{n+1} = -id_nq_n \quad (66)$$

which may be iterated immediately to give the solution

$$d_{n+1} = (-i)^n \frac{d_0}{q_n} \quad (67)$$

where  $d_0$  is the parameter introduced to describe the initial off-centred wavefunction:  $\psi(x, 0) \sim \exp[-\eta x^2/2b^2 + d_0x/b]$ .

Our present interest will not so much be in  $\{\sigma_n\}$  and  $\{d_n\}$ , but rather in the expectation values of the position and momentum. Explicitly evaluating the appropriate expectation values using equation (64) we find

$$\bar{x}_n \equiv \int dx \psi^* x \psi = b \frac{(d_n + d_n^*)}{(\sigma_n + \sigma_n^*)} \quad (68)$$

and

$$\bar{p}_n \equiv \int dx \psi^* (-i\hbar\partial_x) \psi = -\frac{i\hbar}{b} \frac{(\sigma_n^* d_n - \sigma_n d_n^*)}{(\sigma_n + \sigma_n^*)}. \quad (69)$$

It is convenient to re-express the above relations in terms of  $\{q_n\}$  and  $\{d_n\}$  which, due to equation (67), amounts to expressing the expectation values purely in terms of  $\{q_n\}$  and  $d_0$ . Using the known relations (37), (53) and (67), we have

$$\bar{x}_n = \frac{i^n b}{2\eta^R} [q_n d_0^* + (-1)^n q_n^* d_0] \quad (70)$$

and

$$\bar{p}_n = \frac{i^n \hbar}{2b\eta^R} [q_n d_0^* + (-1)^n q_n^* d_0 + i q_{n-1} d_0^* - i(-1)^n q_{n-1}^* d_0]. \quad (71)$$

We are now in a position to utilize the linear iteration rule (41) for the  $\{q_n\}$ . Taking neighbouring differences between the expectation values of the position and momenta, we may use (41) and the definition  $b = (\hbar\tau/m)^{1/2}$  to derive

$$\bar{x}_{n+1} - \bar{x}_n = \frac{\tau}{m} \bar{p}_{n+1} \quad (72)$$

and

$$\bar{p}_{n+1} - \bar{p}_n = -\lambda \bar{x}_n. \quad (73)$$

A direct comparison between these iteration rules with those given in equation (5) for the classical system reveals that the *mean motion* of the off-centred Gaussian wavefunction in the PHP is identical to the purely classical motion—a non-trivial example of Ehrenfest's theorem [14].

Finally, we shall consider the expectation value of the energy. Using the form for the wavefunction given in equation (64) we have (cf equation (25))

$$E_n = \frac{\hbar}{2\tau} \left[ \frac{|\sigma_n|^2}{(\sigma_n + \sigma_n^*)} - \frac{(\sigma_n^* d_n - \sigma_n d_n^*)^2}{(\sigma_n + \sigma_n^*)^2} \right]. \quad (74)$$

Referring to equation (28), we see that the first term on the right-hand side is precisely the 'quantum' energy for the centred Gaussian, studied in section 3. Referring to equation (69), the second term on the right-hand side is determined as nothing more than the classical energy  $\bar{p}_n^2/2m$  studied in section 2. Thus, the energy of the off-centred Gaussian falls neatly into two separate pieces: a 'classical' energy determined purely by the expectation value of the momentum, and a 'quantum' energy determined purely by the internal fluctuations of the wavepacket in the co-moving frame. This ends the study of a quantum particle in a PHP.

## 6. Directed polymer in PHP

Quantum mechanical Green functions may be rewritten as path integrals. It is within this formalism that one may appreciate the close mathematical connection between quantum processes and the statistical mechanics of directed lines. We shall use this connection to discuss the physics of a directed line in thermal equilibrium with a PHP, which is physically realized by a set of static, planar (or sparse [12]), harmonic potentials.

We begin by writing the Feynman path integral [13] for the Green function of a quantum mechanical particle in a potential  $V(x, t)$  (where we restrict our attention to one dimension for simplicity):

$$G(x, t; x_0, t_0) = \int_{y(t_0)=x_0}^{y(t)=x} \mathcal{D}y(s) \exp \left\{ \frac{i}{\hbar} \int_{t_0}^t ds \left[ \frac{m}{2} \left( \frac{dy}{ds} \right)^2 - V(y(s), s) \right] \right\}. \quad (75)$$

This is to be compared with the (restricted) partition function for a directed line in thermal equilibrium with a static potential  $V(x, z)$ . By ‘directed line’ we mean a connected path in a two-dimensional space  $(x, z)$ , which may adopt any configuration whatever, so long as it is directed along the longitudinal ( $z$ ) direction. The statistical mechanics of such objects is of interest in several fields, including directed polymers [16], and superconducting flux lines [17]. The partition function for a line with one end pinned at  $(x_0, z_0)$  and the other pinned at  $(x, z)$  is given by

$$Z(x, z; x_0, z_0) = \int_{y(z_0)=x_0}^{y(z)=x} \mathcal{D}y(s) \exp \left\{ -\frac{1}{T} \int_{z_0}^z ds \left[ \frac{\kappa}{2} \left( \frac{dy}{ds} \right)^2 + V(y(s), s) \right] \right\} \quad (76)$$

where  $T$  is temperature (with Boltzmann’s constant set to unity) and  $\kappa$  is the elasticity of the line.

The obvious similarity between these two path integrals can be misleading, since it is important to remember that the relation between them exists at a strictly mathematical level. Given the analytic solution to one path integral, one may infer the solution to the other by an analytic continuation from ‘real time’  $t$  to ‘imaginary time’  $z$ . However, the physical properties of the two systems are generally quite distinct and little may be inferred about the physics of one of the systems, if only qualitative information about the physics of the other is available. This will become clear in the present case of a PHP, as we shall soon see.

The physical meaning of a directed line in equilibrium with a PHP is as follows. The PHP itself consists of potentials which only exist on discrete transverse lines, and are regularly spaced along the longitudinal axis. These potentials are harmonic and centred at  $x = 0$ . The directed line (which we shall take to be pinned at  $(0, 0)$  and to have a length  $z$ ) equilibrates itself in these potentials, meaning that the free energy is minimized as a result of the competition between the wandering of the line (entropy), and its elastic and potential energies. One may find an application within the field of superconductivity. In an array of flux lines, one may construct an approximate (harmonic) ‘caging’ potential for a given flux line, by averaging over the repulsive line–line interactions which it experiences with its neighbours. Furthermore, in the strongly layered cuprates [18] (which form the most important class of high temperature superconductors) the supercurrents only exist in well separated Cu–O planes. Thus the flux line (meaning the imaginary line connecting the planar centres of magnetic flux) will only experience the caging potential in discrete, but regular, transverse planes.

Given the relationship between the quantum system and the directed line at the level of path integrals, there is naturally a partial differential equation for  $Z$  corresponding to the Schrödinger equation. Defining a ‘rigidity’  $\nu = T/2\kappa$ , and absorbing  $T$  into the potential, we have

$$\partial_z Z = \nu \partial_x^2 Z - V(x, z)Z. \quad (77)$$

The potential is taken to be a PHP, expressed as

$$V(x, z) = \frac{gx^2}{2} \sum_{n=1}^{\infty} \delta(z - nd). \quad (78)$$

The initial condition implicit in the path integral (76) is  $Z(x, 0) = \delta(x)$ , but we may generalize this to any desired function. Following our earlier work on the quantum system, we shall take a Gaussian initial condition

$$Z(x, 0) = \exp[-x^2/a^2] \quad (79)$$



which would naturally arise from thermal wandering of a directed line from a  $\delta$ -function initial condition. The normalization of  $Z$  deserves mention. Whereas in the quantum system, the wavefunction  $\psi$  is normalized by requiring that  $\int dx |\psi|^2 = 1$ , the partition function has an arbitrary prefactor, as we only require that  $\int dx \mathcal{P}(x, z) = 1$ , where  $\mathcal{P}(x, z) = Z(x, z) / \int dx' Z(x', z)$  is the probability density of the line.

We shall not enter into any details concerning the analysis of this system, as our results may be easily reconstructed from the methods presented in section 3 for the quantum analogue. The present system is described by two dimensionless parameters: an effective coupling

$$\tilde{g} = 2vdg \tag{80}$$

and the ratio

$$\chi = 4vd/a^2 = (l/a)^2 \tag{81}$$

which is the square of the ratio of the transverse thermal wandering scale  $l = (4vd)^{1/2}$  and the initial transverse scale  $a$ . The probability density of the line just prior to the  $n$ th impulse is defined as

$$\mathcal{P}_n(x) = \lim_{\epsilon \rightarrow 0} \mathcal{P}(x, nd - \epsilon) \tag{82}$$

and has the explicit form

$$\mathcal{P}_n(x) = \left[ \frac{(1 - q_{n-1}/q_n)}{\pi l^2} \right]^{1/2} \exp \left[ - \left( \frac{q_n - q_{n-1}}{q_n} \right) \left( \frac{x}{l} \right)^2 \right]. \tag{83}$$

This Gaussian form is completely described by one quantity; namely, the width  $\gamma_n$  of the probability density defined via  $\mathcal{P}_n \sim \exp[-x^2/\gamma_n^2]$ . Thus

$$\gamma_n = l \left( \frac{q_n}{q_n - q_{n-1}} \right)^{1/2}. \tag{84}$$

The determinants  $\{q_n\}$  satisfy the second-order difference equation

$$q_{n+2} = \beta q_{n+1} - q_n \tag{85}$$

where  $\beta = \tilde{g} + 2$ , and the initial data are  $q_0 = 1$  and  $q_1 = \chi + 1$ .

Let us first study the case of  $\beta > 2$ , which corresponds to an attractive (or binding) PHP with  $\tilde{g} > 0$ . In this case it is convenient to define a parameter  $\theta$  via

$$\cosh \theta = \beta/2 = 1 + \tilde{g}. \tag{86}$$

Then we find

$$q_n = \frac{1}{\sinh \theta} [\sinh(n + 1)\theta + (\chi + 1 - 2 \cosh \theta) \sinh n\theta]. \tag{87}$$

Substituting this solution into equation (84) yields the final result for the transverse line scale as a function of  $n = z/d$ :

$$\gamma_n = l \left[ \frac{(\chi + 1 - \cosh \theta) \tanh n\theta + \sinh \theta}{(\cosh \theta - 1)(2 \cosh \theta - \chi) \tanh n\theta + (\chi - 2(\cosh \theta - 1)) \sinh \theta} \right]^{1/2}. \tag{88}$$

For the limit of an infinitely long line,  $n\theta \rightarrow \infty$ , the function  $\tanh n\theta \rightarrow 1$ , and the above result simplifies dramatically to

$$\gamma_\infty = \frac{l}{\sqrt{\zeta}} \tag{89}$$

with finite  $n$  corrections  $\sim e^{-2n\theta}$ . The dimensionless parameter  $\zeta$  is given by

$$\zeta \equiv 1 + \sinh \theta - \cosh \theta = 1 - e^{-\theta}. \tag{90}$$

Note that for

$$\zeta = \chi \tag{91}$$

the transverse scale is asymptotically equal to the initial scale  $a$ , meaning that a specially tuned PHP can exactly compensate the transverse wandering for arbitrarily long lines. The tuned value of the coupling  $\tilde{g}$  needed to satisfy equation (91) is

$$\tilde{g} = \frac{\chi^2}{1 - \chi}. \tag{92}$$

Since the coupling  $\tilde{g}$  must be positive, we see that such a compensating PHP is only possible for  $\chi < 1$ , i.e. for a line whose initial scale  $a > l$ . This is clear since whatever the PHP strength, the line is free to wander a transverse scale  $l$  between pulses, and thus we can never restrict the line to  $\gamma_\infty = a$  if  $a < l$ .

Regardless of the initial transverse scale  $a$ , a very strong PHP is expected to strongly compress the line to have a scale  $\gamma_\infty \sim l$ . For  $\tilde{g} \gg 1$ , one can perform an asymptotic expansion on equation (89) to find

$$\gamma_\infty = l \left( 1 + \frac{1}{4\tilde{g}} + \dots \right). \tag{93}$$

The case of a repulsive (or unbinding) PHP is a little more subtle as the line will become unstable (meaning the probability density becomes unnormalizable) beyond a critical length depending on the strength of the potential. In fact, it is easy to see that for the line to survive just one pulse, we require  $\tilde{g} > -2$ . It is therefore convenient to define a parameter  $\theta'$  via

$$\cos \theta' = 1 - |\tilde{g}|/2. \tag{94}$$

We find for the determinants

$$q_n = \frac{1}{\sin \theta'} [\sin(n + 1)\theta' + (\chi + 1 - 2 \cos \theta') \sin n\theta'] \tag{95}$$

and using equation (84),

$$\gamma_n = l \left[ \frac{(\chi + 1 - \cos \theta') \tan n\theta' + \sin \theta'}{(\cos \theta' - 1)(2 \cos \theta' - \chi) \tan n\theta' + (\chi - 2(\cos \theta' - 1)) \sin \theta'} \right]^{1/2}. \tag{96}$$

This expression is not valid for arbitrarily large  $n$ . There exists a maximum length  $n^*d$  for the directed line, beyond which it no longer exists as a connected elastic structure. The value  $n^*$  may be found by demanding that  $q_n > 0$  for all  $n \leq n^*$ . Referring to equation (95) we find that  $n^* = [w^*]$ , where

$$w^* = \frac{1}{\theta'} \left[ \frac{\pi}{2} + \tan^{-1} \left( \frac{\chi + 1 - \cos \theta'}{\sin \theta'} \right) \right]. \tag{97}$$

For  $|\tilde{g}| \rightarrow 2$ ,  $\theta' \rightarrow \pi/2$  and  $1 < w^* < 2$  as expected.

The more interesting limit of  $|\tilde{g}| \rightarrow 0$  yields the result

$$w^* \sim \frac{\pi}{|\tilde{g}|^{1/2}} - \frac{1}{\chi} + O(|\tilde{g}|^{1/2}). \tag{98}$$

The transverse scale of the density may also be studied in the limit of small  $|\tilde{g}|$ . Referring to equation (96) we find  $\gamma_n \sim l\sqrt{n}$  for  $n\theta' \ll 1$ , which is pure thermal wandering. As  $n$  increases further, the potential starts to have an effect, and for  $n\theta' \simeq \pi/2$ , we find that the transverse scale increases linearly with line length  $\gamma_n \sim ln$ , with a prefactor depending in a non-trivial way on  $\chi$ .

## 7. Conclusions

In this paper we have studied in detail the action of a pulsed harmonic potential on three systems: a classical particle, a quantum particle, and a directed line. The first and second systems share some properties via their mechanics, whereas the second and third share a common mathematical basis via the path integral formalism. The pulsing was taken to be regular with a period  $\tau$  (or a longitudinal wavelength  $d$  in the case of a directed line).

The classical particle was studied in section 2. It was found to have stable (or bounded) dynamics as long as the dimensionless coupling  $\xi$  (cf equation (7)) lies in the range  $0 \leq \xi \leq 4$ . Otherwise the motion is unstable. For  $\xi < 0$ , the particle is accelerated to  $|x| = \infty$ , whilst for  $\xi > 4$ , the particle ‘ping-pongs’ with ever increasing amplitude away from the origin. In the stable band, there is either periodic or quasiperiodic motion. We introduced two classes of periodic motion: PMI( $n$ ), for motion where all physical quantities have period  $n\tau$ ; and PMII( $n$ ), for motion where the energy has period  $n\tau$ . We found that the condition for PMI( $n$ ) motion was that  $\phi = M\pi/n$  where  $\phi$  is a convenient parameter defined as  $\cos \phi = 1 - \xi/2$ , and  $M$  is a positive *even* integer satisfying  $M \leq [n/2]$ . Thus the simplest PMI motion occurs for  $n = 3$  and  $M = 2$  (corresponding to  $\xi = 3$ ). PMII( $n$ ) motion was found to occur for  $\phi = M'\pi/n$  with  $M'$  an integer (even or odd) in the range  $1 \leq M' \leq [n/2]$ . The simplest PMII motion occurs for  $n = 2$  and  $M' = 1$  (corresponding to  $\xi = 2$ ). These periodic motions are independent of the initial position  $x_1$  and initial momentum  $p_1$ . We also found one special PMI(2) motion which requires  $\xi = 4$  and a specially tuned initial condition  $p_1 = 2mx_1/\tau$ .

The dynamical properties of a quantum particle in a PHP were studied in sections 3–5. The first two sections concentrated on a Gaussian wavepacket centred at the origin; whilst section 5 pertained to the case of an off-centred Gaussian wavepacket, which has non-zero expectation values of position and momentum, and may therefore be compared directly to the classical particle studied in section 2.

In section 3 we studied the centred Gaussian wavepacket in a PHP using the iteration properties of the determinants of  $n$ -fold Gaussian integrals. This led to second-order linear iteration rules similar to those found in section 2. The stability band for the wavepacket is  $0 \leq \xi < 4$  as in the classical case. If  $\xi < 0$ , the wavepacket is stretched more with each pulse and the width monotonically diverges with time. If  $\xi > 4$ , the wavepacket is squeezed so tightly after each pulse that the velocity of expansion of the width of the packet grows ever greater after each pulse (due to quantum uncertainty). Within the stable band, there are periodic and quasiperiodic dynamics. In the former, the cycles are periodic for all physical properties, thus there is no classification into PMI and PMII as in the classical case. The condition for a cycle of period  $n\tau$  is  $\phi = M\pi/n$  with  $M$  an integer (even or odd) in the range  $1 \leq M \leq [n/2]$ , where  $\phi$  is defined via  $\cos \phi = 1 - \xi/2$  as used in the classical case. The energy portraits for the classical and quantum particles are shown in figures 1–5, and show several distinct features, especially with regard to the upper and lower bounding curves.

In section 4 we analysed the centred Gaussian wavepacket using Fourier methods, which resulted in a first-order, but nonlinear, iteration rule. We showed its equivalence to the second-order linear iteration rule of section 3. This first-order rule allowed two new aspects of the problem to be analysed with ease. The first is the existence of a special cycle of period  $\tau$ , which is a purely quantum mechanical effect, as a classical system must have a cycle of at least  $2\tau$ . This 1-cycle exists only for an initial complex Gaussian wavefunction with an inverse variance tuned to the harmonic coupling via  $\bar{\eta} = 2 \sin(\phi/2) \exp(i\phi/2)$ . An example of a 1-cycle is shown in figure 5. The second aspect is the behaviour of the

system as the time  $\tau$  between pulses is taken to zero. We found that if the initial wave function is chosen to be the ground state of a static harmonic potential (with oscillator frequency  $\omega = (\lambda/m\tau)^{1/2}$ ), then this wavefunction is a fixed point of the PHP dynamics as  $\tau \rightarrow 0$ . This result shows that a pulsed potential with arbitrarily small period cannot mimic a static potential, for even though the ground state wavefunction is a fixed point of the PHP dynamics, there is no phase evolution as would be found for a wavefunction in the static potential. The PHP continually resets the phase clock. This causes concern with regard to numerical integration of the time-dependent Schrödinger equation, where one may discretize time, in which case one is implicitly modelling a static potential by a pulsed potential.

In section 5 we allowed the Gaussian wavepacket to be off-centre, which allowed there to be an evolution of the expectation values  $\bar{x}$  and  $\bar{p}$  of the position and momentum respectively. Using our previous iteration rules in tandem, we showed that these expectation values obeyed the same difference equations as the classical position and momentum, as studied in section 2, thus verifying Ehrenfest's theorem in a non-trivial setting. Furthermore, we evaluated the energy of the wavepacket, and found that it split neatly into two pieces: a 'quantum piece' equal to the energy of the centred Gaussian wavepacket, and a 'classical piece' equal to  $\bar{p}^2/2m$ .

We moved away from the quantum PHP, and in section 6 studied the statistical mechanics of a directed line in a harmonic potential which exists only on discrete transverse lines which are regularly spaced in the longitudinal direction. This system is the imaginary time analogue of the quantum system, as is clear from the path-integral formalism. Using the same methods as in section 3, we found that this system has two qualitatively different regimes, depending on the dimensionless coupling  $\tilde{g}$  (which is the analogous quantity to  $\xi$  as used in the mechanical systems). For  $\tilde{g} > 0$ , we found that the transverse fluctuations of the line saturate rapidly to  $l/\sqrt{\zeta}$ , where  $l$  is the transverse thermal wandering scale between pulses, and  $\zeta = 1 - e^{-\theta}$ , where  $\cosh \theta = 1 + \tilde{g}$ . For a very large attractive coupling, the transverse scale saturates at  $l$  with  $O(1/\tilde{g})$  corrections. For  $\tilde{g} < 0$  the line has a maximum length  $n^*d$  beyond which it is destroyed (as a connected elastic entity) by the repulsive PHP. An exact expression was derived for  $n^*$  (as given in equation (97)), which has the asymptotic form for  $|\tilde{g}| \rightarrow 0$ :  $n^* = [w^*]$ , with  $w^* \sim \pi/|\tilde{g}|^{1/2}$ . Prior to the line breaking up, the transverse wandering grows diffusively for  $n|\tilde{g}|^{1/2} \ll 1$ , and linearly with  $n$  for  $n|\tilde{g}|^{1/2} \simeq \pi/2$ .

These results show clearly the subtle differences which exist between the mechanics of the classical and quantum particles in a PHP; and also the differences which exist between the quantum and statistical path integral expressions for the PHP. In the former case we have seen that the quantum system has another level of complexity beyond the mean (or classical) motion. The periodic motion of the centred Gaussian wavefunction has no classical counterpart, and its energy portrait certainly deserves more study. In the latter case of the quantum versus statistical path integrals, we have seen how no vestige of the periodic behaviour of the quantum system remains in the physics of the directed line. It has a much simpler asymptotic ( $n \rightarrow \infty$ ) behaviour, since all the interesting quantum effects are here damped exponentially in  $n$ . This serves as a warning that one can only retrieve complete quantum information from an imaginary time path integral, if one has complete analytic information.

We believe that these results may also be of some practical interest. It is well known how to trap single particles in specially prepared potentials formed from external magnetic fields [19]. Thus it is possible to make the trapping potential time-dependent by externally varying these fields. It would be of interest to use such external variation to mimic a PHP,

and to test the generality of the results obtained here; namely, the stability band, and its associated periodic and quasi-periodic dynamics.

As detailed in section 6, one can find applications for the directed line in a PHP in the field of superconductivity. Of more recent interest in this field is the role of disorder in layered materials [20], and its efficacy in pinning flux lines. Disorder is a notoriously difficult effect to describe analytically, and there are essentially no analytically solvable cases of directed lines in a quenched disorder potential. We consider the generalization of the regularly pulsed PHP to one with random pulsing intervals (along with a localized columnar pin) to be a prime candidate for such a solvable system, using the framework developed in this paper.

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